

A note on the sample complexity of the Er-SpUD algorithm by Spielman, Wang and Wright for exact recovery of sparsely used dictionaries

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Abstract

We consider the problem of recovering an invertible $n \times n$ matrix A and a sparse $n \times p$ random matrix X based on the observation of $Y = AX$ (up to a scaling and permutation of columns of A and rows of X). Using only elementary tools from the theory of empirical processes we show that a version of the Er-SpUD algorithm by Spielman, Wang and Wright with high probability recovers A and X exactly, provided that $p \geq Cn \log n$, which is optimal up to the constant C .

1 Introduction

Learning sparsely-used dictionaries has recently attracted considerable attention in connection to applications in machine learning, signal processing or computational neuroscience. In particular, two important fields of applications are *dictionary learning* [9, 5, 2, 10, 15] and *blind source separation* [16, 4]. We do not discuss these applications and refer the Reader to the aforesaid articles for details.

Among many approaches to this problem a particularly successful one has been presented by Spielman, Wang and Wright [11, 12], who considered the noiseless-invertible case:

The main problem:

Consider an invertible $n \times n$ matrix A and a random $n \times p$ sparse matrix X . Denote $Y = AX$. The objective is to reconstruct A and X (up to scaling and permutation of columns of A and rows of X) based on the observable data Y .

The Authors of [12] provide an algorithm which with high probability successfully recovers the matrices A and X up to rescaling and permutation of the columns of A and rows of X , provided that X is a sparse random matrix satisfying the following probabilistic assumptions.

Probabilistic model specification

$$X_{ij} = \chi_{ij} R_{ij},$$

where

- χ_{ij}, R_{ij} are independent random variables,

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- χ_{ij} are Bernoulli distributed: $\mathbb{P}(\chi_{ij} = 1) = 1 - \mathbb{P}(\chi_{ij} = 0) = \theta$,
- R_{ij} are i.i.d., with mean zero and satisfy

$$\mu := \mathbb{E}|R_{ij}| \geq 1/10,$$

$$\forall t > 0 \quad \mathbb{P}(|R_{ij}| \geq t) \leq 2e^{-t^2/2}.$$

Following [12] we will say that matrices satisfying the above assumptions follow the Bernoulli-Subgaussian model with parameter θ .

We remark that the constant $1/10$ above is of no importance and has been chosen following [12] and [7].

The approach of Spielman, Wang and Wright consists of two steps. At the first step (given by the Er-SpUD algorithm we describe below) one gathers $p/2$ candidates for the rows of X . The second, greedy step (Greedy algorithm, also described below) selects from the candidates the set of n sparsest vectors, which form a matrix of rank n .

The algorithms work as follows:

ER-SpUD(DC): Exact Recovery of Sparsely-Used Dictionaries using the sum of two columns of Y as constraint vectors.

1. Randomly pair the columns of Y into $p/2$ groups $g_j = \{Ye_{j_1}, Ye_{j_2}\}$.
2. For $j = 1, \dots, p/2$
Let $r_j = Ye_{j_1} + Ye_{j_2}$, where $g_j = \{Ye_{j_1}, Ye_{j_2}\}$.
Solve $\min_w \|w^T Y\|_1$ subject to $r_j^T w = 1$, and set $s_j = w^T Y$.

Above we use the convention that if $r_j = 0$ (which happens with nonzero probability), and as a consequence the minimization problem has no solution, then we skip the corresponding step of the algorithm.

The second stage, described below, is run on the set S of vectors s_i returned at the first stage (for notational simplicity we relabel them if $r_j = 0$ for some j). We use the standard notation that $\|x\|_0$ denotes the number of nonzero coordinates of a vector x .

Greedy: A Greedy Algorithm to Reconstruct X and A .

1. REQUIRE: $S = \{s_1, \dots, s_T\} \subseteq \mathbb{R}^p$.
2. For $i = 1, \dots, n$
REPEAT
 $l \leftarrow \operatorname{argmin}_{s_l \in S} \|s_l\|_0$, breaking ties arbitrarily
 $x_i = s_l$, $S = S \setminus \{s_l\}$
UNTIL $\operatorname{rank}([x_1, \dots, x_i]) = i$
3. Set $X = [x_1, \dots, x_n]^T$ and $A = YY^T(XY^T)^{-1}$.

In [12] it was proved that there exist positive constants C, α , such that if

$$\frac{2}{n} \leq \theta \leq \frac{\alpha}{\sqrt{n}}$$

and $p \geq Cn^2 \log^2 n$, then the ER-SpUD algorithm successfully recovers the matrices A, X with probability at least $1 - \frac{1}{Cp^{10}}$. Note that the equation $Y = A'X'$ still holds if we set $A' = A\Pi\Lambda$ and $X' = \Lambda^{-1}\Pi^T X$ for some permutation matrix Π and a nonsingular diagonal matrix Λ . Therefore, by recovery we mean that nonzero multiples of all the rows of X are among the set $\{s_1, \dots, s_{p/2}\}$ produced by the ER-SpUD(DC) algorithm. In [12] it is also proved that if $\mathbb{P}(R_{ij} = 0) = 0$, then for $p > Cn \log n$, with probability $1 - C'n \exp(-c\theta p)$ for any matrices A', X' such that $Y = A'X'$ and $\max_i \|e_i^T X'\|_0 \leq \max_i \|e_i^T X\|_0$ there exists a permutation matrix Π and a nonsingular diagonal matrix Λ such that $A' = A\Pi\Lambda$, $X' = \Lambda^{-1}\Pi^T X$. In fact, the Authors of [12] prove that with the above probability any row of X is nonzero and has at most $(10/9)\theta p$ nonzero entries, whereas any linear combination of two or more rows of X has at least $(11/9)\theta p$ entries.

In particular it follows that the Greedy algorithm will extract from the set $\{s_1, \dots, s_T\}$ multiples of all n rows of X (note that all s_j 's are in the row space of Y and thus also in the row space of X). Since, as one can prove, X is with high probability of rank n , one easily proves that one can recover A by the formula used in the 3rd step of the algorithm. We remark that in [7] Luh and Vu obtained the same results concerning sparsity of linear combinations of rows of X without the assumptions about the symmetry of the variables R_{ij} .

Note also that for θ of the order n^{-1} , $p = Cn \log n$ is necessary for uniqueness of the solution in the sense described above, otherwise with significant probability some of the rows of X may be zero, which means that some columns of A do not influence the matrix Y .

In [12] it was also proved that if $p > Cn \log n$, $\theta > C'\sqrt{\frac{\log n}{n}}$, then with high probability the ER-SpUD algorithm does not recover any of the rows of X .

Spielman, Wang and Wright have conjectured that their algorithm works with high probability provided that $p > Cn \log n$ (which, as mentioned above is required for well-posedness of the problem).

Recently, Luh and Vu [7] have proved that the algorithm works for $p > Cn \log^4 n$, which differs from the conjectured number of samples just by a polylogarithmic factor.

In this note we will consider a modified version of the algorithm with a slightly different first stage. Namely, instead of using only $p/2$ pairs of columns of Y , we will use all $\binom{p}{2}$ pairs. For fixed p it clearly increases the time complexity of the algorithm (which however remains polynomial), but the advantage of this modification is the possibility of proving that it requires only $p = Cn \log n$ to recover X and A with high probability, which as explained above is optimal. More specifically, we will consider the following algorithm.

Modified ER-SpUD(DC): Exact Recovery of Sparsely-Used Dictionaries using the sum of two columns of Y as constraint vectors.

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For  $i = 1, \dots, p-1$ 
  For  $j = i+1, \dots, p$ 
    Let  $r_{ij} = Ye_i + Ye_j$ 
    Solve  $\min_w \|w^T Y\|_1$  subject to  $r_{ij}^T w = 1$ , and set  $s_{ij} = w^T Y$ .

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The final step of the recovery algorithm is again a greedy selection of the sparsest vectors among the candidates collected at the first step. As before, under the assumption $\mathbb{P}(R_{ij} = 0) = 0$, the greedy procedure successfully recovers X and A , provided that multiples of all the rows of X are present among the input set S .

The main result of this note is

Theorem 1.1. *There exist absolute constants $C, \alpha \in (0, \infty)$ such that if*

$$\frac{2}{n} \leq \theta \leq \frac{\alpha}{\sqrt{n}}$$

and X follows the Bernoulli-Subgaussian model with parameter θ , then for $p \geq Cn \log n$, with probability at least $1 - 1/p$ the modified ER-SpUD algorithm successfully recovers all the rows of X , i.e. multiples of all the rows of X are present among the vectors s_{ij} returned by the algorithm.

Remark Very recently in [13], Sun, Qing and Wright proposed an algorithm with polynomial sample complexity, which recovers well conditioned dictionaries under the assumption that the variables R_{ij} are i.i.d. standard Gaussian and $\theta \leq 1/2$, thus allowing for the first time for a linear number of nonzero entries per column of the matrix X . Their novel approach is based on non-convex optimization. The sample complexity of the algorithms in [13] is however higher than for the Er-SpUD algorithm; as mentioned by the Authors, numerical simulations suggest that it is at least $p = \Omega(n^2 \log n)$ even in the case of orthogonal matrix A . The Authors of [13] conjecture that algorithms with sample complexity $p = O(n \log n)$ should be possible also for large θ .

2 Proof of Theorem 1.1

We will follow the general approach presented in [12] and [7]. The main new part of the argument is an improved bound on the sample complexity for empirical approximation of first moments of arbitrary marginals of the columns of the matrix X , given in Proposition 2.1 below. So as not to reproduce technical and lengthy parts of the original proof, we organize this section as follows. First, we present the crucial Proposition 2.1 and provide a brief discussion of its mathematical content. Next, we present an overview of the main steps in the proof scheme of [12]. For parts of the proof not related to Proposition 2.1 or to the modification of the algorithm considered here, we only indicate the relevant statements from [12], while for the part involving the use of Proposition 2.1 and for the conclusion of the proof we provide the full argument. Proposition 2.1 is proved in Section 3.

Below by e_1, \dots, e_N we will denote the standard basis in \mathbb{R}^N for various choices of N (in particular for $N = n$ and $N = p$). The value of N will be clear from the context and so this should not lead to ambiguity.

By B_1^n we will denote the unit ball in the space ℓ_1^n , i.e. $B_1^n = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$, where for $x = (x(1), \dots, x(n))$, $\|x\|_1 = \sum_{i=1}^n |x(i)|$. The coordinates of a vector x will be denoted by $x(i)$ or if it does not interfere with other notation (e.g. for indexed families of vectors) simply by x_i . Again, the meaning of the notation will be clear from the context.

Proposition 2.1. *Let $U_1, U_2, \dots, U_p, \chi_1, \dots, \chi_p$ be independent random vectors in \mathbb{R}^n . Assume that for some constant M and all $1 \leq i \leq p$, $1 \leq j \leq n$,*

$$\mathbb{E} e^{|U_i(j)|/M} \leq 2 \quad (1)$$

and

$$\mathbb{P}(\chi_i(j) = 1) = 1 - \mathbb{P}(\chi_i(j) = 0) = \theta.$$

Define the random vectors Z_1, \dots, Z_p with the equality $Z_i(j) = U_i(j)\chi_i(j)$ for $1 \leq i \leq p$, $1 \leq j \leq n$ and consider the random variable

$$W := \sup_{x \in B_1^n} \left| \frac{1}{p} \sum_{i=1}^p (|x^T Z_i| - \mathbb{E}|x^T Z_i|) \right|. \quad (2)$$

Then, for some universal constant C and every $q \geq \max(2, \log n)$,

$$\|W\|_q \leq \frac{C}{p} (\sqrt{p\theta q} + q)M \quad (3)$$

and as a consequence

$$\mathbb{P}\left(W \geq \frac{Ce}{p}(\sqrt{p\theta q} + q)M\right) \leq e^{-q}. \quad (4)$$

The above proposition can be considered a quantitative version of the uniform law of large numbers for linear functionals $x^T Z$ indexed by the unit sphere in the space ℓ_1^n . As such it is a classical object of study in the theory of empirical processes. The proof we give uses only Bernstein's inequality (see e.g. [14]) and Talagrand's contraction principle [6], which in a somewhat similar context was applied e.g. in [8, 1].

Let us also remark that in the above proposition we do not require independence between components of the random vectors U_i or χ_i for fixed i , but just independence between the random vectors $U_i, \chi_i, i = 1, \dots, p$.

2.1 Main steps of the proof of Theorem 1.1

As announced, we will now present an outline of the proof of Theorem 1.1, indicating which steps differ from the original argument in [12].

Step 1. A change of variables.

Recall that r_{ij} are sums of two columns of the matrix Y . At the first step of the proof, instead of looking at the original optimization problem

$$\text{minimize } \|w^T Y\|_1 \text{ subject to } r_{ij}^T w = 1 \quad (5)$$

one performs a change of variables $z = A^T w$, $b_{ij} = A^{-1} r_{ij}$, arriving at the optimization problem

$$\text{minimize } \|z^T X\|_1 \text{ subject to } b_{ij}^T z = 1. \quad (6)$$

Note that one cannot solve (6) since it involves the unknown matrices X and A . The goal of the subsequent steps is to prove that with probability separated from zero the solution z_* of (6) is a multiple of one of the basis vectors e_1, \dots, e_n , say $z_* = \lambda e_k$. This means that $w_*^T Y = z_*^T X = \lambda e_k^T X$, i.e. (5) recovers the k -th row of X up to scaling.

Step 2. The solution z_* satisfies $\text{supp}(z_*) \subseteq \text{supp}(b_{ij})$.

At this step we prove the following lemma, which is a counterpart of Lemma 11 in [12]. It is weaker in that we do not consider arbitrary vectors b_{ij} , but only sums of two distinct columns of X (which is enough for the application in the proof of Theorem 1.1). On the other hand it works already for $p > Cn \log n$ and not for $p > Cn^2 \log n$ as the original lemma from [12].

Lemma 2.2. *For $1 \leq i < j \leq p$, define $b_{ij} = X e_i + X e_j$, $I_{ij} = (\text{supp } X e_i) \cup (\text{supp } X e_j)$. There exist numerical constants $C, \alpha > 0$ such that if $2/n \leq \theta \leq \alpha/\sqrt{n}$ and $p > Cn \log n$, then with probability at least $1 - p^{-2}$ the random matrix X has the following property:*

(P1) *For every $1 \leq i < j \leq p$ either $|I_{ij}| \in \{0\} \cup (1/(8\theta), n]$ or every solution z_* to the optimization problem (6) satisfies $\text{supp } z_* \subseteq I_{ij}$.*

To prove the above lemma, one first shows a counterpart of Lemma 16 in [12].

Lemma 2.3. *For any $1 \leq j \leq p$, if $Z = (\chi_{1j} R_{1j}, \dots, \chi_{nj} R_{nj})$, then for all $v \in \mathbb{R}^n$,*

$$\mathbb{E}|v^T Z| \geq \frac{\mu}{8} \sqrt{\frac{\theta}{n}} \|v\|_1.$$

Proof. Let $\varepsilon_1, \dots, \varepsilon_n$ be a sequence of i.i.d. Rademacher variables, independent of Z . By standard symmetrization inequalities (see e.g. Lemma 6.3. in [6]),

$$\mathbb{E}|v^T R| = \mathbb{E} \left| \sum_{i=1}^n v_i \chi_{ij} R_{ij} \right| \geq \frac{1}{2} \mathbb{E} \left| \sum_{i=1}^n v_i \varepsilon_i \chi_{ij} R_{ij} \right|.$$

The random variables $\varepsilon_i R_{ij}$ are symmetric and $\mathbb{E}|\varepsilon_i R_{ij}| = \mu$, so by Lemma 16 from [12], the right-hand side above is bounded from below by $\frac{\mu}{8} \sqrt{\frac{\theta}{n}} \|v\|_1$. \square

The next lemma is an improvement of Lemma 17 in [12], which is crucial for obtaining Lemma 2.2.

Lemma 2.4. *There exists an absolute constant C , such that the following holds for $p > Cn \log n$. Let $S \subseteq \{1, \dots, p\}$ be a fixed subset of size $|S| < \frac{p}{4}$. Let X_S be the submatrix of X , obtained by a restriction of X to the columns indexed by S . With probability at least $1 - p^{-8}$, for any $v \in \mathbb{R}^n$,*

$$\|v^T X\|_1 - 2\|v^T X_S\|_1 > \frac{p\mu}{32} \sqrt{\frac{\theta}{n}} \|v\|_1.$$

Proof. Note first that by increasing the set S , we increase $\|v^T X_S\|_1$, so without loss of generality we can assume that $|S| = \lfloor p/4 \rfloor$. Apply Proposition 2.1 with the vectors $U_j = (R_{1j}, \dots, R_{nj})$ and $\chi_j = (\chi_{1j}, \dots, \chi_{nj})$ and $q = 8 \log p$. Note that our integrability assumptions on R_{ij} imply (1) with M being a universal constant. Therefore, for some absolute constant C and $p \geq Cn \log n$, with probability at least $1 - p^{-8}$ we have

$$\begin{aligned} \sup_{v \in B_1^n} \left| \|v^T X\|_1 - \mathbb{E} \|v^T X\|_1 \right| &\leq C(\sqrt{p\theta \log p} + \log p) \leq 2C\sqrt{p\theta \log p}, \\ \sup_{v \in B_1^n} \left| \|v^T X_S\|_1 - \mathbb{E} \|v^T X_S\|_1 \right| &\leq 2C\sqrt{p\theta \log p}, \end{aligned}$$

where we used that for C sufficiently large, $p/\log p \geq n \geq 1/\theta$.

Thus, by homogeneity, for all $v \in \mathbb{R}^n$,

$$\begin{aligned} \left| \|v^T X\|_1 - \mathbb{E} \|v^T X\|_1 \right| &\leq 2C\sqrt{\theta p \log p} \|v\|_1, \\ \left| \|v^T X_S\|_1 - \mathbb{E} \|v^T X_S\|_1 \right| &\leq 2C\sqrt{\theta p \log p} \|v\|_1. \end{aligned}$$

In particular this means that (using the notation of Proposition 2.1)

$$\begin{aligned} \|v^T X\|_1 &\geq \mathbb{E} \|v^T X\|_1 - 2C\sqrt{\theta p \log p} \|v\|_1 = p\mathbb{E}|v^T Z_1| - 2C\sqrt{\theta p \log p} \|v\|_1, \\ 2\|v^T X_S\|_1 &\leq 2\mathbb{E} \|v^T X_S\|_1 + 4C\sqrt{\theta p \log p} \|v\|_1 = 2|S|\mathbb{E}|v^T Z_1| + 4C\sqrt{\theta p \log p} \|v\|_1, \end{aligned}$$

and so

$$\|v^T X\|_1 - 2\|v^T X_S\|_1 \geq (p - 2|S|)\mathbb{E}|v^T Z_1| - 6C\sqrt{\theta p \log p} \|v\|_1.$$

Now, by Lemma 2.3 and the assumed bound on the cardinality of S , we get

$$\|v^T X\|_1 - 2\|v^T X_S\|_1 \geq \left(\frac{p\mu}{16} \sqrt{\frac{\theta}{n}} - 6C\sqrt{\theta p \log p} \right) \|v\|_1 > \frac{p\mu}{32} \sqrt{\frac{\theta}{n}} \|v\|_1$$

for $p > C'n \log n$, where C' is another absolute constant. \square

We are now in position to prove Lemma 2.2.

Proof of Lemma 2.2. We will show that for each $1 \leq i < j \leq p$ the probability that $0 < |I_{ij}| \leq 1/(8\theta)$ and there exists a solution to (6) not supported on I_{ij} is bounded from above by $1/p^4$. By the union bound over all $i < j$, this implies the lemma.

Fix i, j and let $S = \{l \in [p] : \exists_{k \in I_{ij}} X_{kl} \neq 0\}$. Denote by \mathcal{F}_1 the σ -field generated by Xe_i and Xe_j . Then $\mathcal{A} = \{0 < |I_{ij}| \leq 1/(8\theta)\} \in \mathcal{F}_1$. By independence, for each $k \notin \{i, j\}$, on the event \mathcal{A} ,

$$\mathbb{P}(k \in S | \mathcal{F}_1) \leq 1 - (1 - \theta)^{|I_{ij}|} \leq 1 - e^{-2\theta|I_{ij}|} \leq 1 - e^{-\frac{1}{4}} < \frac{1}{4},$$

where the second inequality holds if α is sufficiently small.

Thus, by independence of columns of X and Hoeffding's inequality,

$$\mathbb{P}\left(|S \setminus \{i, j\}| \leq \frac{p}{4} \middle| \mathcal{F}_1\right) \geq 1 - e^{-cp} \quad (7)$$

for some universal constant $c > 0$. Let z_* be any solution of (6) and denote by z_0 its orthogonal projection on $\mathbb{R}^{I_{ij}} = \{x \in \mathbb{R}^n : x_k = 0 \text{ for } k \notin I_{ij}\}$. Set also $z_1 = z_* - z_0$ and let X_S, X_{S^c} be the matrices obtained from X by selecting the columns labeled by S and $S^c = [p] \setminus S$ respectively. By the triangle inequality, and the fact that $z_0^T X_{S^c} = 0$, we get

$$\begin{aligned} \|z_*^T X\|_1 &= \|(z_0^T + z_1^T)X_S\|_1 + \|(z_0^T + z_1^T)X_{S^c}\|_1 \\ &\geq \|z_0^T X_S\|_1 - \|z_1^T X_S\|_1 + \|z_1^T X\|_1 - \|z_1^T X_S\|_1 \\ &= \|z_0^T X\|_1 + (\|z_1^T X\|_1 - 2\|z_1^T X_S\|_1). \end{aligned}$$

Denote now by X' the $|I_{ij}^c| \times (p-2)$ matrix obtained by restricting X to the rows from I_{ij}^c and columns from $[p] \setminus \{i, j\}$. Set also $S' = S \setminus \{i, j\}$. If, slightly abusing the notation, we identify z_1 with a vector from $\mathbb{R}^{|I_{ij}^c|}$, we have

$$\|z_1^T X\|_1 - 2\|z_1^T X_S\|_1 = \|z_1^T X'\|_1 - 2\|z_1^T X'_{S'}\|_1,$$

where we used the fact that $z_1^T X e_i = z_1^T X e_j = 0$.

Denote by \mathcal{F}_2 the σ -field generated by Xe_i, Xe_j and the rows of X labeled by I_{ij} (note that I_{ij} is itself random, but this will not be a problem in what follows). The random set S is measurable with respect to \mathcal{F}_2 . Moreover, due to independence and identical distribution of the entries of X , conditionally on \mathcal{F}_2 the matrix X' still follows the Bernoulli-Subgaussian model with parameter θ . Therefore, by Lemma 2.4, if C is large enough, then on $\{|S'| \leq p/4\}$ we have

$$\mathbb{P}\left(\text{for all } v \in \mathbb{R}^{|I_{ij}^c|} : \|v^T X'\|_1 - 2\|v^T X'_{S'}\|_1 \geq \frac{p\mu}{32} \sqrt{\frac{\theta}{|I_{ij}^c|}} \|v\|_1 \middle| \mathcal{F}_2\right) \geq 1 - p^{-8}.$$

Note that by the definition of z_0 , we have $b_{ij}^T z_0 = b_{ij}^T z_* = 1$, therefore z_0 is a feasible candidate for the solution of the optimization problem (6). Thus, we have $\|z_1^T X'\|_1 - 2\|z_1^T X'_{S'}\|_1 \leq 0$ and as a consequence, on the event $\{|S'| \leq p/4\}$,

$$\mathbb{P}(\text{for some solution } z_* \text{ to (6), } z_1 \neq 0 | \mathcal{F}_2) \leq p^{-8}. \quad (8)$$

Thus, denoting $\mathcal{B} = \{\text{for some solution } z_* \text{ to (6), } z_1 \neq 0 \text{ and } 0 < |I_{ij}^c| < 1/(8\theta)\}$, we get by (7) and (8),

$$\begin{aligned} \mathbb{P}(\mathcal{B}) &\leq \mathbb{P}(\mathcal{B} \cap \{|S'| > p/4\}) + \mathbb{E}\mathbb{P}(\mathcal{B} | \mathcal{F}_2) \mathbf{1}_{\{|S'| \leq p/4\}} \\ &\leq \mathbb{E}\mathbb{P}(|S'| > p/4 | \mathcal{F}_1) \mathbf{1}_{\mathcal{A}} + p^{-8} \\ &\leq e^{-cp} + p^{-8} \leq p^{-4} \end{aligned}$$

for $p > Cn \log n$ with a sufficiently large absolute constant C . \square

Step 3. With high probability $z_* = \lambda e_k$ for $k = \operatorname{argmax}_{1 \leq l \leq n} |b_{ij}(l)|$.

At this step one proves the following lemma (Lemma 12 in [12]). Since no changes with respect to the original argument are required (we do not use Proposition 2.1 here), we do not reproduce the proof and refer the Reader to [12] for details. We remark that although the lemma is formulated in [12] for symmetric variables, the symmetry assumption is not used in its proof.

Below, by $|b|_1^\downarrow \geq |b|_2^\downarrow \geq \dots \geq |b|_n^\downarrow$, we denote the nonincreasing rearrangement of the sequence $|b_1|, \dots, |b_n|$, while for $J \subseteq [n]$, X^J denotes the matrix obtained from X by selecting the rows indexed by the set J .

Lemma 2.5. *There exist two positive constants c_1, c_2 such that the following holds. For any $\gamma > 0$ and $s \in \mathbb{Z}_+$, such that $\theta s < \gamma/8$ and p such that*

$$p \geq \max \left\{ \frac{c_1 s \log n}{\theta \gamma^2}, n \right\}, \quad \text{and} \quad \frac{p}{\log p} \geq \frac{c_2}{\theta \gamma^2},$$

with probability at least $1 - 4p^{-10}$, the random matrix X has the following property.

(P2) *For every $J \subseteq [n]$ with $|J| = s$ and every $b \in \mathbb{R}^s$, satisfying $\frac{|b|_1^\downarrow}{|b|_2^\downarrow} \leq 1 - \gamma$, the solution to the restricted problem*

$$\text{minimize } \|z^T X^J\|_1 \text{ subject to } b^T z = 1, \tag{9}$$

is unique, 1-sparse, and is supported on the index of the largest entry of b .

Step 4. Conclusion of the proof.

Set $s = 12\theta n + 1$. Our first goal is to prove that with probability at least $1 - 1/p^2$, for all $k \in [n]$, there exist $i, j \in [p]$, $i \neq j$ such that the vector $b = Xe_i + Xe_j$ satisfies the assumptions of Lemma 2.5, $|b|_1^\downarrow = |b_k|$ and $I_{ij} := (\operatorname{supp} Xe_i) \cup (\operatorname{supp} Xe_j)$ satisfies $0 < |I_{ij}| \leq 1/(8\theta)$, which will allow us to take advantage of Lemma 2.2.

Note that we have

$$\mathbb{E} R_{ij}^2 \leq 4 \int_0^\infty t e^{-t^2/2} dt = 4.$$

Since $\mathbb{E}|R_{ij}| = \mu \geq \frac{1}{10}$, by the Paley-Zygmund inequality (see e.g. Corollary 3.3.2. in [3]), we have

$$\mathbb{P}(|R_{ij}| \geq \frac{1}{20}) \geq \frac{3}{4} \frac{(\mathbb{E}|R_{ij}|)^2}{\mathbb{E} R_{ij}^2} \geq c_0$$

for some universal constant $c_0 > 0$. In particular $\mathbb{P}(|R_{ij}| = 0) < 1 - \frac{c_0}{2}$. Let q be any $(1 - c_0/(2s))$ -quantile of $|R_{ij}|$, i.e. $\mathbb{P}(|R_{ij}| \leq q) \geq (1 - c_0/(2s))$ and $\mathbb{P}(|R_{ij}| \geq q) \geq c_0/(2s)$. In particular, since $s \geq 1$, we get $q > 0$. We have $\mathbb{P}(R_{ij} \geq q) \geq c_0/(4s)$ or $\mathbb{P}(R_{ij} \leq -q) \geq c_0/(4s)$. Let us assume that $\mathbb{P}(R_{ij} \geq q) \geq c_0/(4s)$, the other case is analogous.

Define the event \mathcal{E}_{ki} as

$$\mathcal{E}_{ki} = \left\{ \chi_{ki} = 1, |\{r \in [n] \setminus \{k\} : \chi_{ri} = 1\}| \leq (s-1)/2, R_{ki} \geq q, \forall_{r \neq k} \chi_{ri} = 1 \implies |R_{ri}| \leq q \right\}$$

We will assume that $p \geq 2Cn \log n$ for some numerical constant C to be fixed later on. For $k \in [n]$, consider the events

$$\mathcal{A}_k = \bigcup_{1 \leq i \leq \lfloor p/2 \rfloor} \mathcal{E}_{ki}$$

and

$$\mathcal{B}_i = \bigcup_{1 \leq i \leq \lfloor p/2 \rfloor} \bigcup_{\lfloor p/2 \rfloor < j \leq p} \mathcal{E}_{ki} \cap \mathcal{E}_{kj} \cap \left\{ \{l \in [n] : \chi_{li} = \chi_{lj} = 1\} = \{k\} \right\}.$$

We will first show that for all $k \in [n]$,

$$\mathbb{P}(\mathcal{A}_k) \geq 1 - \frac{1}{p^4}, \quad (10)$$

which we will use to prove that

$$\mathbb{P}(\mathcal{B}_k) \geq 1 - \frac{1}{p^3}. \quad (11)$$

Let us start with the proof of (10). Set $\mathcal{B}_{ki} = \{|\{r \in [n] \setminus \{k\} : \chi_{rk} = 1\}| \leq (s-1)/2\}$. By independence we have

$$\begin{aligned} \mathbb{P}(\mathcal{E}_{ki}) &= \mathbb{P}(\chi_{ki} = 1) \mathbb{P}(R_{ki} \geq q) \mathbb{P}(\mathcal{B}_{ki}) \mathbb{P}(\forall_{r \neq k} \chi_{ri} = 1 \implies |R_{ri}| \leq q | \mathcal{B}_{ki}) \\ &\geq \theta \frac{c_0}{4s} \left(1 - \frac{2\theta(n-1)}{s-1}\right) \left(1 - \frac{c_0}{2s}\right)^{(s-1)/2}, \end{aligned}$$

where to estimate $\mathbb{P}(\mathcal{B}_{ki})$ we used Markov's inequality. The right hand side above is bounded from below by c_1/n for some universal constant c_1 . Therefore if the constant C is large enough, we obtain

$$\mathbb{P}\left(\bigcap_{1 \leq i \leq \lfloor p/2 \rfloor} \mathcal{E}_{ki}^c\right) \leq \left(1 - \frac{c_1}{n}\right)^{\lfloor p/2 \rfloor} \leq \exp(-c_1 p / (4n)) \leq \exp(-4 \log p) = \frac{1}{p^4},$$

where we used the inequality $p / \log p \geq 16c_1^{-1}n$ for $p \geq Cn \log n$. We have thus established (10).

Let us now pass to (11). Denote by \mathcal{F}_1 the σ -field generated by $\chi_{ki}, R_{ki}, k \in [n], 1 \leq i \leq \lfloor p/2 \rfloor$.

For $\omega \in \mathcal{A}_k$ define $i_{\min}(\omega) = \min\{1 \leq i \leq \lfloor p/2 \rfloor : \omega \in \mathcal{E}_{ki}\}$. Note that on \mathcal{A}_k ,

$$\mathbb{P}(\mathcal{B}_k | \mathcal{F}_1) \geq \mathbb{P}\left(\bigcup_{\lfloor p/2 \rfloor < j \leq p} (\mathcal{E}_{kj} \cap \left\{\{l \in [n] : \chi_{li_{\min}} = \chi_{lj} = 1\} = \{k\}\right\}) \middle| \mathcal{F}_1\right)$$

Define

$$\mathcal{C}_{kj} = \{|\{r \in [n] \setminus \{k\} : \chi_{rj} = 1\}| \leq (s-1)/2\} \cap \left\{\{l \in [n] : \chi_{li_{\min}} = \chi_{lj} = 1\} = \{k\}\right\}.$$

Similarly as in the argument leading to (10), for fixed j , using the independence of the variables χ_{lm}, R_{lm} we obtain

$$\begin{aligned} &\mathbb{P}\left(\mathcal{E}_{kj} \cap \left\{\{l \in [n] : \chi_{li_{\min}} = \chi_{lj} = 1\} = \{k\}\right\} \middle| \mathcal{F}_1\right) \\ &= \mathbb{P}(R_{kj} \geq q) \mathbb{E}\left(\mathbf{1}_{\mathcal{C}_{kj}} \mathbb{P}(\forall_{r \neq k} \chi_{rj} = 1 \implies |R_{rj}| \leq q | \mathcal{C}_{kj}, \mathcal{F}_1) \middle| \mathcal{F}_1\right) \\ &\geq \mathbb{P}(R_{kj} \geq q) \mathbb{E}\left(\mathbf{1}_{\mathcal{C}_{kj}} \left(1 - \frac{c_0}{2s}\right)^{\frac{s-1}{2}} \middle| \mathcal{F}_1\right) \\ &= \mathbb{P}(R_{kj} \geq q) \left(1 - \frac{c_0}{2s}\right)^{\frac{s-1}{2}} \mathbb{P}(\mathcal{C}_{kj} | \mathcal{F}_1) \\ &\geq \frac{c_0}{4s} \left(1 - \frac{c_0}{2s}\right)^{\frac{s-1}{2}} \times \\ &\quad \left(\mathbb{P}\left(\{l \in [n] : \chi_{li_{\min}} = \chi_{lj} = 1\} = \{k\} \middle| \mathcal{F}_1\right) - \mathbb{P}\left(\chi_{kj} = 1, |\{r \in [n] \setminus \{k\} : \chi_{rj} = 1\}| > \frac{s-1}{2} \middle| \mathcal{F}_1\right)\right) \\ &\geq \frac{c_0}{4s} \left(1 - \frac{c_0}{4s}\right)^{\frac{s-1}{2}} \left(\theta(1-\theta)^{(s-1)/2} - \theta \frac{2\theta(n-1)}{s-1}\right). \end{aligned}$$

Now recall that $\theta \leq \frac{\alpha}{n}$ for some universal constant α . If α is small enough then $1 - \theta \geq e^{-2\theta}$ and

$$(1 - \theta)^{(s-1)/2} \geq e^{-\theta(s-1)} = e^{-12\theta^2 n} \geq e^{-12\alpha^2} \geq \frac{1}{3}.$$

Since $\frac{2\theta(n-1)}{s-1} \leq \frac{1}{6}$, this implies that

$$\mathbb{P}\left(\mathcal{E}_{kj} \cap \left\{\{l \in [n]: \chi_{lj_{\min}} = \chi_{lj} = 1\} = \{k\}\right\} \middle| \mathcal{F}_1\right) \geq \frac{c_2}{n}$$

for some positive universal constant c_2 . Since the events $\mathcal{E}_{kj} \cap \left\{\{l \in [n]: \chi_{lj_{\min}} = \chi_{lk} = 1\} = \{k\}\right\}$, $\lfloor p/2 \rfloor < k \leq p$ are conditionally independent, given \mathcal{F}_1 , we obtain that on \mathcal{A}_k ,

$$\mathbb{P}(\mathcal{B}_k^c | \mathcal{F}_1) \leq \left(1 - \frac{c_2}{n}\right)^{\lfloor p/2 \rfloor} \leq \frac{1}{p^4},$$

provided C is a sufficiently large universal constant. Now, using (10), we get

$$\mathbb{P}(\mathcal{B}_k) \geq \mathbb{E} \mathbf{1}_{\mathcal{A}_k} \mathbb{P}(\mathcal{B}_k) \geq \mathbb{P}(A_k) \left(1 - \frac{1}{p^4}\right) \geq \left(1 - \frac{1}{p^4}\right)^2 \geq 1 - \frac{1}{p^3},$$

proving (11).

Taking the union bound over $k \in [n]$, we get

$$\mathbb{P}\left(\bigcap_{1 \leq k \leq n} \mathcal{B}_k\right) \geq 1 - \frac{1}{p^2}.$$

Set $\gamma = 1/2$ and observe that if C is large enough and α small enough, then the assumptions of Lemma 2.2 and Lemma 2.5 are satisfied. Moreover $s = 12\theta n + 1 \leq \frac{1}{8\theta}$. Recall the properties P1 and P2 considered in the said lemmas. Consider the event $\mathcal{A} = \bigcap_{1 \leq k \leq n} \mathcal{B}_k \cap \{\text{properties P1 and P2 hold}\}$ and note that $\mathbb{P}(\mathcal{A}) \geq 1 - \frac{1}{p}$. On the event \mathcal{A} , for every k , there exist $1 \leq i < j \leq p$, such that

- $1 \leq |I_{ij}| \leq s \leq 1/(8\theta)$,
- the largest entry of b (in absolute value) equals $b_k \geq 2q > 0$ whereas the remaining entries do not exceed q ,

In particular, by property P1 we obtain that any solution z_* to the problem (6) satisfies $\text{supp } z_* \subseteq I_{ij}$. Therefore for some (any) $J \supseteq I_{ij}$ with $|J| = s$, we obtain (identifying vectors supported on J with their restrictions to J), that z_* is in fact a solution to the restricted problem (9) with $b = b_{ij}$, which by property P2 implies that $z_* = \lambda e_k$ for some $\lambda \neq 0$.

According to the discussion at the beginning of Step 1, this means that the solution w_* to (5) satisfies $w_*^T Y = \lambda e_k^T X$, i.e. the algorithm, when analyzing the vector b_{ij} , will add a multiple of the k -th row of X to the collection S .

This ends the proof of Theorem 1.1.

3 Proof of Proposition 2.1

The first tool we will need is the classical Bernstein's inequality (see e.g. Lemma 2.2.11 in [14]).

Lemma 3.1 (Bernstein's inequality). *Let Y_1, \dots, Y_p be independent mean zero random variables such that for some constants M, v and every integer $k \geq 2$, $\mathbb{E}|Y_i|^k \leq k! M^{k-2} v/2$. Then, for every $t > 0$,*

$$\mathbb{P}\left(\left|\sum_{i=1}^p Y_i\right| \geq t\right) \leq 2 \exp\left(-\frac{t^2}{2(pv + Mt)}\right).$$

As a consequence, for every $q \geq 2$,

$$\left\|\sum_{i=1}^p Y_i\right\|_q \leq C(\sqrt{qp v} + qM), \tag{12}$$

where C is a universal constant.

Another (also quite standard) tool we will rely on is the contraction principle for empirical processes due to Talagrand (see Theorem 4.12. in [6]).

Lemma 3.2 (Talagrand's contraction principle). *Let $F: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be convex and increasing. Let further $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a 1-Lipschitz function such that $\varphi(0) = 0$. For every bounded subset T of \mathbb{R}^n , if $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d. Rademacher variables, then*

$$\mathbb{E}F\left(\sup_{t \in T} \frac{1}{2} \left| \sum_{i=1}^n \varphi(t_i) \varepsilon_i \right| \right) \leq \mathbb{E}F\left(\sup_{t \in T} \left| \sum_{i=1}^n t_i \varepsilon_i \right| \right)$$

Proof of Proposition 2.1. Let $\varepsilon_1, \dots, \varepsilon_p$ be i.i.d. Rademacher variables, independent of the sequences (U_i) , (χ_i) . By the symmetrization inequality (see e.g. Lemma 6.3. in [6]) we have

$$\mathbb{E}W^q \leq 2^q \mathbb{E} \sup_{x \in B_1^n} \left| \frac{1}{p} \sum_{i=1}^p \varepsilon_i |x^T Z_i| \right|^q.$$

Now, since the function $t \mapsto |t|$ is a contraction, an application of Lemma 3.2, conditionally on Z_i , gives

$$\begin{aligned} \mathbb{E}W^q &\leq 2^{2q} \mathbb{E} \sup_{x \in B_1^n} \left| \frac{1}{p} \sum_{i=1}^p \varepsilon_i x^T Z_i \right|^q = \frac{2^{2q}}{p^q} \mathbb{E} \sup_{x \in B_1^n} \left| x^T \sum_{i=1}^p \varepsilon_i Z_i \right|^q \\ &= \frac{2^{2q}}{p^q} \mathbb{E} \left\| \sum_{i=1}^p \varepsilon_i Z_i \right\|_\infty^q = \frac{2^{2q}}{p^q} \mathbb{E} \max_{1 \leq j \leq n} \left| \sum_{i=1}^p \varepsilon_i Z_i(j) \right|^q \\ &\leq \frac{2^{2q}}{p^q} \sum_{j=1}^n \mathbb{E} \left| \sum_{i=1}^p \varepsilon_i Z_i(j) \right|^q. \end{aligned} \tag{13}$$

Now, for every i, j and every integer $k \geq 2$ we have

$$\mathbb{E}|Z_i(j)|^k = \theta \mathbb{E}|U_i(j)|^k \leq \theta M^k k! \mathbb{E}e^{|U_i(j)|/M} \leq 2k! \theta M^k = k! v M^{k-2}/2$$

with $v = 4\theta M^2$. Thus by the moment version (12) of Bernstein's inequality for some universal constant C we get

$$\mathbb{E} \left| \sum_{i=1}^p \varepsilon_i X_i(j) \right|^q \leq C^q \left(\sqrt{qp\theta} M + qM \right)^q,$$

which, when combined with (13), yields for $q \geq \log n$,

$$\|W\|_q \leq \frac{4Ce}{p} (\sqrt{p\theta q} + q)M.$$

The first part of the proposition follows by adjusting the constant C . The tail bound is a direct consequence of the Chebyshev inequality for the q -th moment. \square

References

- [1] R. Adamczak, A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann, *Quantitative estimates of the convergence of the empirical covariance matrix in log-concave ensembles*, J. Amer. Math. Soc. **23** (2010), no. 2, 535–561. 5
- [2] A. M. Bruckstein, D. L. Donoho, and M. Elad, *From sparse solutions of systems of equations to sparse modeling of signals and images*, SIAM Rev. **51** (2009), no. 1, 34–81. 1

- [3] Víctor H. de la Peña and Evarist Giné, *Decoupling*, Probability and its Applications (New York), Springer-Verlag, New York, 1999, From dependence to independence, Randomly stopped processes. *U*-statistics and processes. Martingales and beyond. 8
- [4] P. Georgiev, F. Theis, and A. Cichocki, *Sparse component analysis and blind source separation of underdetermined mixtures*, IEEE Transactions on Neural Networks **16** (2005), no. 4. 1
- [5] K. Kreutz-Delgado, J. Murray, B. Rao, K. Engan, T. Lee, and T. Sejnowski, *Dictionary learning algorithms for sparse representation*, Neural Computation **15** (2003), no. 20, 349–396. 1
- [6] M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 23, Springer-Verlag, Berlin, 1991, Isoperimetry and processes. 5, 6, 11
- [7] K. Luh and V. Vu, *Dictionary Learning with Few Samples and Matrix Concentration*, preprint, <http://arxiv.org/abs/1503.08854> (2015). 2, 3, 4
- [8] S. Mendelson, *On weakly bounded empirical processes*, Math. Ann. **340** (2008), no. 2, 293–314. 5
- [9] B. Olshausen and D. Field, *Emergence of simple-cell receptive field properties by learning a sparse code for natural images*, Nature **381** (1996), no. 6538, 607–609. 1
- [10] R. Rubinstein, A. Bruckstein, and M. Elad, *Dictionaries for sparse representation modeling*, Proceedings of the IEEE **98** (2010), no. 6, 1045–1057. 1
- [11] D. Spielman, H. Wang, and J. Wright, *Exact recovery of sparsely-used dictionaries*, Journal of Machine Learning Research: Workshop and Conference Proceedings **23** (2012), 25th Annual Conference on Learning Theory (COLT). 1
- [12] D. Spielman, H. Wang, and J. Wright, *Exact recovery of sparsely-used dictionaries*, preprint, <http://www.columbia.edu/~jw2966> (2012). 1, 2, 3, 4, 5, 6, 8
- [13] J. Sun, Q. Qing, and J. Wright, *Complete Dictionary Recovery over the Sphere II: Recovery by Riemannian Trust-region Method*, preprint, <http://arxiv.org/abs/1511.04777> (2015). 4
- [14] A. W. van der Vaart and J. A. Wellner, *Weak convergence and empirical processes*, Springer Series in Statistics, Springer-Verlag, New York, 1996, With applications to statistics. 5, 10
- [15] J. Yang, J. Wright, T. S. Huang, and Y. Ma, *Image super-resolution via sparse representation*, IEEE Trans. Image Process. **19** (2010), no. 11, 2861–2873. 1
- [16] M. Zibulevsky and B. Pearlmutter, *Blind source separation by sparse decomposition*, Neural Computation **13** (2001), no. 4. 1